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2001 J. Phys. A: Math. Gen. 34 6157

(http://iopscience.iop.org/0305-4470/34/31/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 6157–6166 PII: S0305-4470(01)23451-9

The Polyakov conjecture and the μ-holomorphy equation relationship

M Kachkachi[1,2] **and M Kessabi**[2]

[1] UPM, Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Hassan 1er, Settat, Morocco
[2] UFR-PHE, Département de Physique, Faculté des Sciences, Université MV, Rabat, Morocco

Received 27 March 2001
Published 27 July 2001
Online at stacks.iop.org/JPhysA/34/6157

Abstract

We determine the iterative solution, on the complex plane, of the μ-holomorphy equation. Then we obtain the μ-holomorphic projective connection as a Neumann series in powers of the Beltrami differential μ. Since it has been shown that the Polyakov action of a conformal model with a central charge k is expressed in terms of the μ-holomorphic projective connection, we then prove the Polyakov conjecture.

PACS numbers: 11.25.Hf, 02.40.-k

1. Introduction

The geometrical character of bi-dimensional conformal models has been stressed and expressed in different papers. For literature (dating back 15 years) on the conformal symmetry and its application in bosonic string theories see [4, 5].

Indeed, the fact that a conformal model is related to a Riemann surface Σ gives arise to the fact that the perturbative development that contains the Green functions of the model is expressed in powers of the Beltrami differential μ, $|\mu| < 1$ [6–8]. The latter is interpreted as an exterior source of the energy–momentum tensor associated with a bi-dimensional conformal model of central charge k for which Polyakov has used [9, 10] the following two conjectures: (1) uniqueness of renormalization: the Polyakov action that is a solution of the conformal Ward identity resums the iterative series provided by the renormalization field theory. (2) Universality: this solution is independent of the model's fields.

Geometrically, the Beltrami differentials parametrize the complex structures on the Riemann surface Σ on which the model is constructed. Due to this fact, the transition from a reference complex structure $(z, \bar{z})$ to another one $(Z(z, \bar{z}); \overline{Z}(z, \bar{z}))$ parametrized by μ, where the conformal invariance is always maintained; $\frac{\partial Z}{\partial \overline{Z}} = 0$ is called a μ-quasiconformal transformation and is defined by the Beltrami equation

$$(\overline{\partial} - \mu\partial)Z = 0$$
$$\partial \equiv \frac{\partial}{\partial z} \qquad \overline{\partial} \equiv \frac{\partial}{\partial \overline{z}}. \tag{1.1}$$

This geometrical formulation of the bi-dimensional conformal model, based on the Beltrami parametrization of complex structures enables one to give an essential role to the locality. The advantage of this functional framework, that does not use a metric on Σ, is that it naturally implies the holomorphic factorization. Then the $Z(z, \overline{z})$ coordinate that parametrizes the μ-complex structure on Σ and that is the solution of the Beltrami equation (1.1) is interpreted as the Wess–Zumino field on which depends the Polyakov action (the effective action of a bi-dimensional conformal model of central charge k) [6]:

$$\Gamma_{\mathrm{WZP}}[\mu] = -\frac{k}{12}\int_C \mathrm{dm}\ (\mathrm{z}, \overline{\mathrm{z}})(\mu\partial^2 \ln \partial Z)(z, \overline{z}) \tag{1.2}$$

with

$$\mathrm{d}m\ (z, \overline{z}) \equiv \frac{\mathrm{d}\overline{z} \wedge \mathrm{d}z}{2\mathrm{i}\pi}.$$

On the other hand, each complex structure is associated with a projective structure that is parametrized by a projective connection R. This latter is required to preserve the conformal covariance of the diffeomorphisms anomaly [2, 3, 9] that appears in the conformal Ward identity [3, 6, 9]:

$$(\overline{\partial} - \mu\partial - 2\partial\mu)\frac{\delta\Gamma_{\mathrm{WZP}}}{\delta\mu} = \frac{k}{12}(\partial^3\mu + \mu\partial R + 2R\partial\mu). \tag{1.3}$$

For example, the projective connection associated with a reference complex structure is holomorphic and is denoted by R_0:

$$\overline{\partial} R_0 = 0. \tag{1.4}$$

However, the projective connection, associated with a μ-projective structure is not holomorphic in the reference complex structure, is denoted by R and satisfies the μ-holomorphic equation [2, 9]:

$$\overline{\partial} R = \partial^3\mu + 2R\partial\mu + \mu\partial R. \tag{1.5}$$

Among the results presented in this paper, we determine the iterative solution of equation (1.5) by giving the corresponding Neumann series with the help of the Cauchy kernel techniques for the $\overline{\partial}$-operator that were developed in [6–8].

2. The Beltrami equation

Let us consider for the sense-preserving diffeomorphism f the derivative $\partial_\alpha f$ in the direction α [10]:

$$\partial_\alpha f = \partial f + \mathrm{e}^{-2\mathrm{i}\alpha}\overline{\partial} f \tag{2.1}$$

where $\partial f \equiv \frac{\partial f(z)}{\partial z}$.

Then we have

$$\max_\alpha |\partial_\alpha f| = |\partial f| + |\overline{\partial} f|$$
$$\min_\alpha |\partial_\alpha f| = |\partial f| - |\overline{\partial} f| \tag{2.2}$$

where $\|$ denotes the absolute value, and the dilatation quotient

$$D_f \equiv \frac{\max_\alpha |\partial_\alpha f|}{\min_\alpha |\partial_\alpha f|} \tag{2.3}$$

is finite. Hence, we can write

$$D_f \leqslant K \tag{2.4}$$

for every $z \in A$. A is a domain on a Riemann surface Σ.

Moreover, the Jacobian $J_f = |\partial f|^2 - |\overline{\partial} f|^2$, for a sense-preserving diffeomorphism f, is positive. Then $\partial f(z) \neq 0$, and we can form the quotient:

$$\mu(z) \equiv \frac{\overline{\partial} f(z)}{\partial f(z)}. \tag{2.5}$$

The function μ, so defined in A, is called the analytic dilatation of f. Since f is continuous, μ is Borel-measurable function, and from (2.4) we see that

$$|\mu(z)| \leqslant \frac{K-1}{K+1} \prec 1. \tag{2.6}$$

The definition of complex dilatation leads us to consider the Beltrami equation

$$\overline{\partial} f = \mu \partial f \tag{2.7}$$

where μ is measurable and $\|\mu\|_\infty \prec 1$.

If f is conformal, μ vanishes identically, and (2.7) becomes the Cauchy–Riemann equation $\overline{\partial} f = 0$.

A sense-preserving diffeomorphism f with the property (2.4) is called a K-quasiconformal mapping.

By definition, a collection ϕ of functions ϕ_α defined on each domain A_α by

$$\begin{aligned} &\phi_\alpha : A_\alpha \to \Sigma \\ &\phi_\alpha = \phi \circ z_\alpha \end{aligned} \tag{2.8}$$

is called a (p, q)-differential on Σ if it is invariant under holomorphic change of coordinates: $(A_\alpha, z_\alpha) \to (A_\beta, z_\beta)$. It is written locally as

$$\phi = \phi_{pq}(z, \overline{z})\, \mathrm{d}z^p\, \mathrm{d}\overline{z}^q. \tag{2.9}$$

As an example, the Beltrami differential μ is $(-1, 1)$-differential:

$$\mu = \mu \frac{z}{z}\, \mathrm{d}\overline{z} \otimes \partial. \tag{2.10}$$

It is interpreted as a C^∞ section of the fibre bundle $k^{-1} \otimes \overline{k}$, where k is the holomorphic cotangent bundle on Σ.

Under a holomorphic change of coordinates $z \to \omega(z)$, we have the following transformation law:

$$\mu \frac{\omega}{\omega} = (\partial\omega)(\overline{\partial\omega})^{-1} \mu \frac{z}{z}. \tag{2.11}$$

In particular, let us consider on Σ two complex variable functions $Z(z, \overline{z})$ (and its c.c.) whose differential 1-form is expressed as

$$\mathrm{d}Z = \lambda(\mathrm{d}z + \mu\, \mathrm{d}\overline{z}) \tag{2.12}$$

where μ is a C^∞-function and satisfies the ellipticity condition as before:

$$|\mu| < 1. \tag{2.13}$$

In this transition from the local reference complex structure $(z, \bar{z})$ to the other one $(Z(z, \bar{z}), \overline{Z}(z, \bar{z}))$ that is parametrized by the Beltrami differential $\mu(z, \bar{z})$, the function $\lambda(z, \bar{z})$ is interpreted as the conformal factor.

As the differential of the two-variables function $\mathrm{d}Z$ is a linear combination of differentials $\mathrm{d}z$, $\mathrm{d}\bar{z}$ whose coefficients are the partial derivatives of the function $Z(z, \bar{z})$, we have

$$\mathrm{d}Z = \partial Z\, \mathrm{d}z + \bar{\partial} Z\, \mathrm{d}\bar{z} \tag{2.14}$$

with

$$\partial Z \equiv \frac{\partial Z}{\partial z} \qquad \bar{\partial} Z \equiv \frac{\partial Z}{\partial \bar{z}}.$$

The combination of equations (2.12) and (2.14) gives the following relations:

$$\partial Z = \lambda \tag{2.15}$$

$$\bar{\partial} Z = \lambda\mu. \tag{2.16}$$

These are equivalent to the partial derivatives equation

$$(\bar{\partial} - \mu\partial) Z = 0 \tag{2.17}$$

which is the Beltrami equation satisfied by new coordinates Z and defines the quasiconformal transformation between the two complex structures $(z, \bar{z})$ and $(Z(z, \bar{z}), \overline{Z}(z, \bar{z}))$. Indeed, if $\mu = 0$ then we have

$$\bar{\partial} Z = 0 \tag{2.18}$$

which is a conformal transformation:

$$(z, \bar{z}) \to (Z(z), \overline{Z}(\bar{z})). \tag{2.19}$$

In other words, the solution of equation (2.17) is interpreted as a deformation of the reference conformal structure by the parameter μ. Then the function $\mu(z, \bar{z})$ corresponds to a new conformal structure whose coordinates system is $(Z(z, \bar{z}), \overline{Z}(z, \bar{z}))$. Hence, each conformal structure corresponds to a Beltrami differential that can be defined by the Beltrami equation. Indeed, from equation (2.17) we get

$$\mu = \frac{\bar{\partial} Z}{\partial Z}. \tag{2.20}$$

3. The Cauchy kernel

Now, let us define on the complex Riemann surface Σ the inverse of the operator $\bar{\partial}$ as follows [6, 10]:

$$\bar{\partial} F(z, \bar{z}) = f(z, \bar{z}) \tag{3.1}$$

where F, $f \in C^{\infty}(\Sigma)$. Then,

$$F(z, \bar{z}) = \int_{\Sigma} \mathrm{d}m\, (\omega, \bar{\omega}) N(\omega, z) f(\omega, \bar{\omega}). \tag{3.2}$$

With

$$\mathrm{d}m\, (\omega, \bar{\omega}) = \frac{\mathrm{d}\omega \wedge \mathrm{d}\bar{\omega}}{2\mathrm{i}\pi}.$$

The function $N(\omega, z)$ is called the specified Cauchy kernel corresponding to the specified Riemann surface. For example, on the complex plane C, the Cauchy kernel is given by [6]

$$N(\omega, z) \equiv \frac{1}{z - \omega} \tag{3.3}$$

and then, the function F is expressed as

$$F(z,\bar{z}) = \int_C \mathrm{d}m\,(\omega,\bar{\omega})\frac{f(\omega,\bar{\omega})}{z-\omega}. \tag{3.4}$$

Now, let us consider the Beltrami equation (2.6) in the following form:

$$\bar{\partial} Z = \mu\partial Z. \tag{3.5}$$

So, the function $Z(z,\bar{z})$ parametrizing the new complex structure that corresponds to the Beltrami differential $\mu(z,\bar{z})$ is given by

$$Z(z,\bar{z}) = \int_\Sigma \mathrm{d}m\,(\omega,\bar{\omega})N(\omega,z)(\mu\partial Z)(\omega,\bar{\omega}). \tag{3.6}$$

Then Z is a (non-local) holomorphic functional of μ as well as the integration factor λ.

4. The iterative solution of the Beltrami equation on C

Let us consider the application of the ∂-operator on equation (3.5):

$$\bar{\partial}\partial Z = \partial\mu\partial Z + \mu\partial^2 Z. \tag{4.1}$$

Then, with the help of equation (2.4), we have

$$\bar{\partial}\lambda = \lambda\partial\mu + \mu\partial\lambda. \tag{4.2}$$

By using the definition

$$\Lambda \equiv \ln\lambda \tag{4.3}$$

equation (4.2) becomes

$$\bar{\partial}\Lambda = \partial\mu + \mu\partial\Lambda \tag{4.4}$$

that is, another form of the Beltrami equation.

4.1. The generalized Cauchy formula

The resolution of the Beltrami equation in terms of powers of μ is performed by inverting the $\bar{\partial}$-operator using the Cauchy kernel techniques [13]: if $f \in D(C)$ with, $D(C) = \{\mu/\mu \in C^\infty(C)$ and μ has a compact support$\}$.

Then we define the inverse of the $\bar{\partial}$-operator on the complex plane by the relation

$$(\bar{\partial}^{-1} f)(\omega) = \int_C \mathrm{d}m\,(z,\bar{z})\frac{f(z)}{\omega - z} \tag{4.5}$$

where $N(\omega,z) = \frac{1}{\omega-z}$ is the Cauchy kernel on C.

Then we have

$$\begin{aligned} f &= \bar{\partial}(\bar{\partial}^{-1} f) \\ &= \bar{\partial}^{-1}(\bar{\partial} f). \end{aligned} \tag{4.6}$$

By using the definition

$$\partial(\bar{\partial}^{-1} f) \equiv \partial\bar{\partial}^{-1} f \tag{4.7}$$

we obtain the following expressions:

$$(\partial\bar{\partial}^{-1} f)(\omega) = \int_C \mathrm{d}m\,(z,\bar{z})\partial_\omega \frac{1}{\omega - z} f(z) \tag{4.8}$$

$$= \int_C \mathrm{d}m\,(z,\bar{z})\partial_z \frac{1}{z-\omega} f(z) \tag{4.9}$$

that are well defined in the sense of distributions because for $f \in D(C)$, $\bar{\partial}^{-1} f$ and $\partial\bar{\partial}^{-1} f$ are C^∞ and out of the support of f they are analytic functions [12].

Then, by using the limit condition $\Lambda_{|\mu=0} = 0$, we can write

$$\partial\Lambda = (\partial\bar{\partial}^{-1})\bar{\partial}\Lambda \tag{4.10}$$

because $\mu \in D(C)$ implies $\bar{\partial}\Lambda \in D(C)$. Then, by substituting equation (4.10) in (4.4), we get the following:

$$\bar{\partial}\Lambda = \partial\mu + \mu(\partial\bar{\partial}^{-1})\bar{\partial}\Lambda. \tag{4.11}$$

4.2. The iterative solution of the Beltrami equation on C

Now, let us consider the following Neumann series:

$$\begin{aligned} g_1 &= \partial\mu \\ g_k &= \mu(\partial\bar{\partial}^{-1})g_{k-1} \qquad k = 2, 3, \ldots \end{aligned} \tag{4.12}$$

and then, we can formally write

$$\bar{\partial}\Lambda = \sum_{k=1}^{+\infty} g_k. \tag{4.13}$$

To express the general term g_k in powers of μ, in the reference coordinates system (0), let us use the notation of [7]:

$$\begin{aligned} z_i &\equiv i \qquad i = 0, 1, 2, \ldots \\ \mathrm{d}m_i &\equiv \frac{\mathrm{d}\bar{z}_i \wedge \mathrm{d}z_i}{2\mathrm{i}\pi} \qquad i = 0, 1, 2, \ldots \\ \partial_i &\equiv \frac{\partial}{\partial z_i} \\ z_{ij} &\equiv z_i - z_j \qquad \text{and} \qquad \mu_i \equiv \mu(z_i). \end{aligned} \tag{4.14}$$

Then, by using the relation (4.8), we get the general term in the reference coordinates as

$$g_k(0) = \int_C \mathrm{d}m_1\, \mu_0 \partial_0 \left(\frac{1}{z_{01}}\right) g_{k-1}(1). \tag{4.15}$$

The insertion of the induction formula (equation (4.12)) into equation (4.15) enables us to write

$$g_k(0) = \int_C \prod_{i=1}^{k-1} \mathrm{d}m_i \prod_{i=0}^{k-2} \partial_i \left(\frac{1}{z_{ii+1}}\right) \partial_k \mu_{k-1} \prod_{i=0}^{k-2} \mu_i. \tag{4.16}$$

Hence, we determine the $\bar{\partial}^{-1}$-operator action on the general terms as follows:

$$\bar{\partial}_0^{-1} g_k(0) = \int_C \prod_{i=0}^{k-1} (\mathrm{d}m_i\, \mu_i) \prod_{i=0}^{k-3} \partial_i \left(\frac{1}{z_{ii+1}}\right) \partial_{k-2}^2 \left(\frac{1}{z_{k-2k-1}}\right). \tag{4.17}$$

Due to the ellipticity condition; $|\mu| < 1$ and $\mu \in C^\infty(C)$, the Neumann series that is constructed is C^∞ in the reference coordinates system (0) and is uniformally convergent.

5. The projective connection

Besides the complex structure on the Reimann surface Σ, there is a projective structure which is parametrized by a projective connection. For example, a holomorphic projective connection R_0 on Σ; $\bar{\partial} R_0 = 0$ is an assignment, to any coordinate z of a reference conformal structure, of a smooth function R_0 defined in the domains of z and z' by the following transformation [12]:

$$R'_{z'z'}(z') = (\partial' z)^2 [R_{zz}(z) - S(z'; z)] \tag{5.1}$$

where

$$S(z'; z) = \partial^2 \ln \partial z' - \tfrac{1}{2} \partial \ln \partial z' \tag{5.2}$$

is the Schwarzian derivative.

Furthermore, there exists a Beltrami differential $\mu = \mu_{\bar{z}}^{z}\, \mathrm{d}\bar{z} \otimes \partial z$ on Σ; that is a $(-1, 1)$-differential and then a C^∞ section of the fibre bundle $K^{-1} \otimes \overline{K}$, where K is a holomorphic cotangent bundle on Σ [14], such that R is μ-holomorphic, i.e.

$$(\bar{\partial} - \mu\partial - 2\partial\mu) R = \partial^3 \mu. \tag{5.3}$$

This means that to any element R of the space of all projective connections satisfying equation (5.3), there is canonically associated a projective structure subordinated to the conformal structure which is parametrized by μ. In particular, one can deduce from equation (5.3) (by putting $\mu = 0$) that, in the reference complex structure, the μ-holomorphic projective connection becomes holomorphic. In another way, a quasiconformal transformation transforms, in a reference complex coordinates system, a holomorphic projective connection into a μ-holomorphic one:

$$R_0(z) \xrightarrow{\mu} R(z, \bar{z}) \qquad \bar{\partial} R_0 = 0 \longrightarrow \bar{\partial} R = (\partial^3 + 2R\partial + \partial R)\mu \equiv L_3(\mu) \tag{5.4}$$

where L_3 is the third Bol's operator [15].

Physically speaking, a holomorphic projective connection R_0 in the reference conformal structure ensures the correct conformal covariance of the right-hand side of the conformal Ward identity [2, 6, 11]:

$$(\bar{\partial} - \mu\partial - 2\partial\mu) \frac{\delta \Gamma[\mu]}{\delta \mu} = \frac{-k}{24\pi} (\partial^3 \mu + 2R_0 \partial\mu + \partial R_0 \mu) \tag{5.5}$$

where $\Gamma[\mu]$ is the effective action of an arbitrary conformal field theory on a Reimann surface Σ (for a chiral sector) when the matter field is integrated out. k is the central charge of the model.

Indeed, the diffeomorphisms anomaly $A(C, \mu) = -\mu L_3(C)$, where C is the ghost field in the BRST formalism [6], transforms with a Jacobian under a conformal change of coordinates:

$$A(C, \mu)' = (\partial' z)(\overline{\partial' z}) A(C, \mu). \tag{5.6}$$

6. The iterative solution of the μ-holomorphy equation

Now, let us write equation (5.3) as follows:

$$\bar{\partial} R = \partial^3 \mu + 2R\partial\mu + \mu\partial R. \tag{6.1}$$

Then by using the inverse of the $\bar{\partial}$-operator (6.1) becomes

$$\bar{\partial} R = \partial^3 \mu + D\bar{\partial} R \tag{6.2}$$

with

$$D \equiv (2\partial\mu + \mu\partial)\bar{\partial}^{-1}. \tag{6.3}$$

Hence, $\bar{\partial}R$ is equal to the Neumann series that has a finite limit for $|\mu| < \varepsilon \leqslant 1$:

$$\bar{\partial}R = \sum_{k=1}^{\infty} g_k \tag{6.4}$$

where

$$g_1 = \partial^3 \mu \tag{6.5}$$

and

$$g_k = D g_{k-1}. \tag{6.6}$$

So, in some coordinate system say, $(z_0) \equiv (0)$, the μ-holomorphic projective connection is given by

$$R(0) = \sum_{k=1}^{\infty} \bar{\partial}_0^{-1} g_k(0). \tag{6.7}$$

The Cauchy kernel formula given before, enables us to get

$$\bar{\partial}_0^{-1} g_k(0) = \int_C \mathrm{d}m_1 \frac{g_k(1)}{z_{01}} \tag{6.8}$$

and by using equation (6.6) we have

$$\bar{\partial}_0^{-1} g_k(0) = \int_C \mathrm{d}m_1 \frac{D_1 g_{k-1}(1)}{z_{01}} \tag{6.9}$$

with

$$D_i = (2\partial_i \mu_i + \mu_i \partial_i)\bar{\partial}_i^{-1}. \tag{6.10}$$

Then we obtain the k-term of the perturbative series in terms of μ of the μ-holomorphic projective connection as follows:

$$\bar{\partial}_0^{-1} g_k(0) = (-1)^k \int_C \prod_{i=1}^{k} (\mathrm{d}m_i\, \mu_i) \partial_k^3 A_{k-1} \tag{6.11}$$

where

$$A_0 \equiv \frac{1}{z_{01}} \tag{6.12}$$

and

$$A_k = \frac{2\partial_k A_{k-1}}{z_{kk+1}} + A_{k-1}\partial_k \left(\frac{1}{z_{kk+1}}\right). \tag{6.13}$$

Finally, the μ-holomorphic projective connection on the complex plane, in some reference coordinates system $(z) \equiv (0)$ is given by

$$R(0) = \sum_{k=1}^{\infty} (-1)^k \int_C \prod_{i=1}^{k} (\mathrm{d}m_i \mu_i) \partial_k^3 A_{k-1}. \tag{6.14}$$

Here, we stress that we recover the results of [2] by putting in equation (6.14) $k = 1, 2, 3$. Then our results generalize those of [2].

Out of the critical dimensions, the Weyl anomaly can by shifted to a diffeomorphism anomaly by extracting from the effective action a suitable local counterterm and this leads to exploitation of the holomorphic feature of the diffeomorphism anomaly [1]. Local forms of this counterterm have been proposed in the literature [1,2] and the references therein. However, the suitable form, on an arbitrary compact Riemann surface without boundary, was given by

Knecht *et al* [1]. They have found that three terms are involved and they have obtained, in the space of local functionals, the following equivalence equation for the s-cohomology:

$$A(\Omega, g) + s[\Gamma_{\mathrm{I}} + \Gamma_{\mathrm{II}} + \Gamma_{\mathrm{III}}] = A(C, \mu) + \overline{A(C, \mu)} \tag{6.15}$$

where $A(\Omega, g)$ is the Weyl anomaly which is a functional of the Weyl ghost Ω and of the metric g on a Riemann surface Σ. s is the BRST operator associated with the diffeomorphism group. $A(C, \mu) + \overline{A(C, \mu)}$ is the chirally split diffeomorphism anomaly which depends on the vector field $C = c + \mu\bar{c}$ (the combination of the diffeomorphism ghosts) and on the Beltrami differential μ.

Γ_{I} is the Liouville action written in terms of a (1, 1)-conformal field in a background which is designed to absorb the Weyl anomaly.

Γ_{II} is the second term that was expressed in terms of the holomorphic and the μ-holomorphic projective connections [2]:

$$\Gamma_{\mathrm{II}} = \frac{k}{12}\int_{\Sigma} \mathrm{d}m_0 \mu_0 (R - R_0)(0) + \mathrm{c.c.} \tag{6.16}$$

Γ_{III} is the third term that completes the elimination of the background to the benefit of the conformal class of metrics. Then, by using the expression of the μ-holomorphic projective connection (equation (6.14)), we get the following form of the action Γ_{II}:

$$\Gamma_{\mathrm{II}} = -\frac{k}{12}\int_C \mathrm{d}m_0\, R_0(0) - \frac{k}{12}\sum_{n=1}^{\infty}(-1)^{n+1}\int_C \prod_{i=0}^{n} (\mathrm{d}m_i\, \mu_i)\partial_n^3 A_{n-1}. \tag{6.17}$$

Furthermore, it was shown in [3] that the Polyakov action for a conformal model is given by

$$\Gamma_{\mathrm{P}} = \Gamma_{\mathrm{II}}[\mu] + 2\int_{\Sigma} \mathrm{d}m_0\, \mu_0 f(0) \tag{6.18}$$

where f is an element of the Cauchy kernel of the Ward operator $\Delta \equiv \bar{\partial} - \mu\partial - 2\partial\mu$.

Then we were able to get the explicit expression, at any order of the perturbation series, for the Polyakov action by using the results of [2, 3, 7, 8], from which all Green functions can by easily derived. This implies that the action (6.17) resums the perturbative series in terms of powers of the parameter μ that is interpreted as the exterior source of the energy–momentum tensor. This is the Neumann series, the solution of the μ-holomorphy equation.

Then we have expressed explicitly the Polyakov conjecture on the complex plane: on C, for each model of central charge k, the formal series

$$\frac{k}{12}\sum_{k=1}^{\infty}(-1)^{k}\int_C \prod_{i=0}^{k} (\mathrm{d}m_i\, \mu_i)\partial_k^3 A_{k-1} \tag{6.19}$$

that appears in equation (6.17) is resumed by the Polyakov action $\Gamma_{\mathrm{P}}[\mu]$ for $\mu \in D(C)$.

7. Conclusion

Here, we give the explicit expression of the μ-holomorphic projective connection that is needed to preserve the conformal covariance in the μ-complex structure and to present the chirally split diffeomorphisms anomaly. Then we prove the Polyakov conjecture on the complex plane with the help of the solution of the μ-holomorphic projective connection equation and of the expression (6.16) of the Polyakov action.

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